

# Extending the Wait-free Hierarchy to Multi-Threaded Systems

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# Introduction

## Special instructions

- ▶ Atomic operations on registers that are added in the hardware
- ▶ Examples: test-and-set, fetch-and-add, compare-and-swap, load-link/store conditional...

## Universality

$O$  (shared object or register with a special instruction) is universal if:

- ▶ Any object with a sequential specification has a linearizable implementation using read/write registers and instances of  $O$

## Wait-freedom

- ▶ Every operation terminates, no matter crashes or asynchrony

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## Wait-freedom

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# The Wait-free Hierarchy

## Consensus number

- ▶ Consensus number  $k \in \mathbb{N}$ :
  - ▶ Wait-free universal for  $k$  processes
  - ▶ Not wait-free universal for  $k + 1$  processes
- ▶ Consensus number  $\infty$ :
  - ▶ Wait-free universal for  $k$  processes, for all  $k$

## Significance

- ▶ Objects with CN  $x$  cannot implement objects with CN  $y > x$

# The Wait-free Hierarchy in Multi-Threaded Systems

## Multi-threaded systems

- ▶ Threads can be created dynamically
  - ▶ No bound on the number of threads in an execution
- ▶ Allocation of unbounded but **finite** arrays
  - ▶ How to allocate one shared register to each thread?

## The iterator stack:

- ▶ Infinite consensus number
- ▶ Not universal in multi-threaded systems

## Problem statement

How to compare the synchronization power of shared objects in multi-threaded systems?

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# Arrival Models

## Maximal number of processes in an execution:

- $M_1^n$  Classical model
  - ▶ At most  $n$  processes (known to the developer)
- $M_1$  Bounded arrival model
  - ▶ The bound is known at initialization
- $M_2$  Finite arrival model
  - ▶ After some time, no new thread is started
- $M_3$  Infinite arrival model
  - ▶ New threads may keep arriving

## Starting an extended hierarchy

	Universal in the arrival model?			
Infinite	X	X	X	✓
Finite	X	X	✓	✓
Bounded	X	✓	✓	✓
Objects	$O_1 < O_2 \dots$	$< O$	$< O'$	$< O''$

Finite consensus number      Infinite consensus number

# Arrival Models

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## Starting an extended hierarchy

	Universal in the arrival model?			
Infinite	✗	✗	✗	✓
Finite	✗	✗	✓	✓
Bounded	✗	✓	✓	✓
Objects	$O_1 < O_2 \dots$	$< O$	$< O'$	$< O''$

Finite consensus number      Infinite consensus number

# Extended Wait-free Hierarchy

			Universal without infinite allocation?					
Arrival Models			Infinite	X	X	X	✓	
			Finite	X	X	✓	✓	
		Infinite	Finite	Bounded	X	✓	✓	✓
Universal with infinite allocation?	✓	✓	✓					
	X	✓	✓					
	X	X	✓					
	X	X	X					

# Extended Wait-free Hierarchy

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Finite				
			Bounded				
Universal with infinite allocation?	✓	✓	✓				
	✗	✓	✓				
	✗	✗	✓				
	✗	✗	✗				

# Extended Wait-free Hierarchy

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Infinite	Finite	Bounded		
			Infinite	Finite	Bounded		
Universal with infinite allocation?	✓	✓	✓	X	X	X	✓
	X	✓	✓	X	X	✓	✓
	X	X	✓	X	✓	✓	✓
	X	X	X	X	✓	✓	✓
							Universal in multi-threaded systems

# Extended Wait-free Hierarchy

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Infinite	Finite	Bounded		
			Infinite	Finite	Bounded		
Universal with infinite allocation?	✓	✓	✓				
	✗	✓	✓				
	✗	✗	✓				
	✗	✗	✗				
							Universal in multi-threaded systems

# Filling the Hierarchy: State of the Art

			Universal without infinite allocation?				
Arrival Models			Infinite	X	X	X	✓
			Finite	X	X	✓	✓
Infinite		Finite	Bounded	X	✓	✓	✓
		Bounded					
Universal with infinite allocation?	✓	✓	✓	?	?	?	?
	X	✓	✓	?	?	?	?
	X	X	✓	?	?	?	?
	X	X	X	?	?	?	?

# Filling the Hierarchy: State of the Art

Arrival Models		Infinite			Universal without infinite allocation?			
					Finite	Finite	Finite	Finite
Infinite	Finite	Bounded	Infinite	Finite	Finite	Finite	Finite	Finite
✗	✓	✓	?	?	?	?	?	?
✗	✗	✓	?	?	?	?	?	?
✗	✗	✗	?	?	?	?	?	?

empty  
(if universal without infinite allocation, still universal with infinite allocation)



# Filling the Hierarchy: State of the Art

		Arrival Models			Universal without infinite allocation?			
					Infinite	Finite	Finite	Finite
		Infinite	Finite	Bounded				
					X	X	X	✓
					X	X	✓	✓
					X	✓	✓	✓
Universal with infinite allocation?	✓	✓	✓		?	?	?	consensus
	X	✓	✓		?	?	?	empty
	X	X	✓		?	?	(if universal without infinite allocation, still universal with infinite allocation)	
	X	X	X		?			

# Filling the Hierarchy: State of the Art

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Finite				
			Bounded				
			Infinite	Finite	Bounded		
Universal with infinite allocation?	✓	✓	✓	?	?	?	consensus
	✗	✓	✓	?	?	iterator stack	empty (if universal without infinite allocation, still universal with infinite allocation)
	✗	✗	✓	?	?		
	✗	✗	✗	?			

# Filling the Hierarchy: State of the Art

Arrival Models		Universal without infinite allocation?						
		Infinite	Finite	Bounded				
Universal with infinite allocation?	Infinite	✓	✓	✓	✗	✗	✗	✓
	Finite	✗	✓	✓	✗	✗	✓	✓
	Bounded	✗	✗	✓	✗	✓	✓	✓
		✗	✗	✗	✗	✗	✗	✗

Universal with infinite allocation?	Infinite			Finite		Bounded	
	✓	✓	✓	?		?	consensus
	✗	✓	✓	?		?	iterator stack
	✗	✗	✓	finite consensus number		?	empty
	✗	✗	✗	(if universal without infinite allocation, still universal with infinite allocation)			

# Filling the Hierarchy: State of the Art

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Finite				
			Bounded				
			Infinite	Finite	Bounded		
Universal with infinite allocation?	✓	✓	✓		?	?	consensus
	✗	✓	✓	empty	?	iterator stack	
	✗	✗	✓		?	empty	
	✗	✗	✗				

Read/Write
T&S
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(if universal without infinite allocation, still universal with infinite allocation)
[Herlihy1991]

# Objects with a Finite Consensus Number

## Proposition

If an object  $O$  is universal in the classical model  $M_1^n$  enriched with infinite arrays, then  $O$  is universal in the classical model  $M_1^n$ .

## Remarks

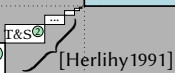
- ▶ The same proof works for the bounded arrival model
- ▶ This does not prove that the two models were equivalent

## Sketch of the proof

- ▶ An algorithm  $A$  uses  $O$  to solve binary consensus
  - ▶  $A$  has a bounded number of inputs (thanks to renaming)
  - ▶ For each input,  $A$  accesses a bounded number of objects
  - ▶ Therefore,  $A$  does not need infinite arrays
- ▶ Binary consensus is universal in the classical model

# Filling the Hierarchy: Consensus Number $\infty_1^1$

Arrival Models		Universal without infinite allocation?			
		Infinite	Finite	Bounded	
Universal with infinite allocation?	Infinite	∞	∞	∞	✓
	Finite	∞	∞	✓	✓
	Infinite	∞	✓	✓	✓
	Finite	∞	✓	✓	✓
✓	✓	✓	empty	$\infty_1^3$ $\infty_2^3$ $\infty_3^3$ consensus	
∞	✓	✓	empty	$\infty_1^2$ $\infty_2^2$ $\infty_3^2$ iterator stack	
∞	∞	✓	empty	$\infty_1^1$ $\infty_2^1$ $\infty_3^1$ empty	
∞	∞	∞	Read/Write	(if universal without infinite allocation, still universal with infinite allocation)	



# Filling the Hierarchy: Consensus Number $\infty_1^1$

			Universal without infinite allocation?				
Arrival Models			Infinite				
			Finite				
			Bounded				
			Infinite	Finite	Bounded		
Universal with infinite allocation?	✓	✓	✓	empty	$\infty_1^3$	$\infty_2^3$	$\infty_3^3$ consensus
	✗	✓	✓		$\infty_1^2$	$\infty_2^2$	$\infty_3^2$
	✗	✗	✓		$\infty_1^1$	$\infty_2^1$	$\infty_3^1$
	✗	✗	✗		Read/Write	T&S	[Herlihy1991]

# No Object has Consensus Number $\infty_1^1$

## Proposition

If an object  $O$  is universal in the bounded arrival model  $M_1$ , then  $O$  is universal in the finite arrival model  $M_2$  enriched with infinite arrays.

## Sketch of the proof

We propose a round-based algorithm. At round  $k$ :

- ▶ Processes  $p_{k+1}, \dots$ 
  - ▶ Mark `greaterId[k]`
  - ▶ If `adopt[k]` was written, adopt the value
  - ▶ Start a new round
- ▶ Processes  $p_1, \dots, p_k$ 
  - ▶ Solve consensus together **using  $O$  in model  $M_1^k$**
  - ▶ Write the decided value in `adopt[k]`
  - ▶ If `greaterId[k]` is marked, start a new round
  - ▶ Otherwise, decide



# Filling the Hierarchy: Consensus Number $\infty_1^2$

			Universal without infinite allocation?							
			$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$				
Arrival Models		Infinite	$\times$	$\times$	$\times$	$\checkmark$				
		Finite	$\times$	$\times$	$\checkmark$	$\checkmark$				
		Bounded	$\times$	$\checkmark$	$\checkmark$	$\checkmark$				
Universal with infinite allocation?	Infinite	Finite	Bounded	empty			$\infty_1^3$	$\infty_2^3$	$\infty_3^3$	consensus
	$\checkmark$	$\checkmark$	$\checkmark$				$\infty_1^2$	$\infty_2^2$	$\infty_3^2$	
	$\times$	$\checkmark$	$\checkmark$	empty			$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	
	$\times$	$\times$	$\checkmark$	empty			$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	
$\times$	$\times$	$\times$	<div style="border: 1px solid black; padding: 2px; display: inline-block;">                     T&amp;S                      ...                      Read/Write                 </div> [Herlihy1991]			(if universal without infinite allocation, still universal with infinite allocation)				

## Window register

- ▶ Size  $k$  chosen at initialization
- ▶ A read returns the  $k$  last values written

# Filling the Hierarchy: Consensus Number $\infty_1^2$

		Universal without infinite allocation?						
		Infinite	Finite	Bounded	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$
Arrival Models		Infinite	Finite	Bounded	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$
		$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$
Universal with infinite allocation?	✓	✓	✓	empty	empty	empty	empty	consensus
	✗	✓	✓	empty	window registers	iterator stack	empty	empty
	✗	✗	✓	empty	empty	empty	empty	empty
	✗	✗	✗	empty	empty	empty	empty	empty

(if universal without infinite allocation, still universal with infinite allocation)

Read/Write [Herlihy1991]

## Window register

- ▶ Size  $k$  chosen at initialization
- ▶ A read returns the  $k$  last values written

# Filling the Hierarchy: Consensus Number $\infty_1^2$

		Universal without infinite allocation?							
		Infinite	Finite	Bounded	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$	
Arrival Models		Infinite	Finite	Bounded	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	$\infty_3^2$	
		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	
		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	
		$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	$\times$	
Universal with infinite allocation?	Yes	$\checkmark$	$\checkmark$	$\checkmark$	empty	?	$\infty_1^3$	$\infty_2^3$	consensus $\infty_3^3$
	No	$\times$	$\checkmark$	$\checkmark$	empty	↑ window registers $\infty_1^2$	iterator stack $\infty_2^2$	$\infty_3^2$	$\infty_3^2$
	No	$\times$	$\times$	$\checkmark$	empty $\infty_1^1$	empty $\infty_2^1$	empty $\infty_3^1$	$\infty_3^1$	$\infty_3^1$
	No	$\times$	$\times$	$\times$	Read/Write $\infty_1^1$	(if universal without infinite allocation, still universal with infinite allocation)			

## Window register

- ▶ Size  $k$  chosen at initialization
- ▶ A read returns the  $k$  last values written

# Filling the Hierarchy: Consensus Number $\infty_1^2$

		Universal without infinite allocation?						
Arrival Models		Infinite	Finite	Bounded				
Infinite	Finite				X	X	X	✓
					X	X	✓	✓
		Infinite	Finite	Bounded				
Universal with infinite allocation?	✓	✓	✓	empty		$\infty_1^3$	$\infty_2^3$	$\infty_3^3$ consensus
	X	✓	✓		window registers	$\infty_1^2$	$\infty_2^2$	$\infty_3^2$
	X	X	✓	empty	$\infty_1^1$	$\infty_2^1$	$\infty_3^1$	
	X	X	X	Read/Write	(if universal without infinite allocation, still universal with infinite allocation)			

*Note: A red arrow points from the 'window registers' cell to the 'iterator stack' cell, and another red arrow points from the 'iterator stack' cell to the 'consensus' cell. A red question mark is also present in the 'window registers' cell.*

## Window register

- ▶ Size  $k$  chosen at initialization
- ▶ A read returns the  $k$  last values written

# Window Registers are not Universal in $M_3$

## Proposition

It is impossible to solve consensus in the infinite arrival model enriched with infinite arrays, using only window registers

## Sketch of the proof

**$n$ -critical configuration** If only  $p_1, \dots, p_n$  take steps, consensus is decided by the next step of a process

- ▶ There are  $n$ -critical configurations for all  $n$
- ▶ In an  $n$ -critical configurations,  $p_1, \dots, p_n$  are about to write in a window register of size at least  $k$
- ▶ We build an infinite execution in which some process passes through  $n$ -critical configurations for all  $n$

# Window Registers need Infinite Arrays in $M_2$

## Proposition

It is impossible to solve consensus in the finite arrival model, using only window registers and binary consensus objects

## Sketch of the proof

- ▶ Suppose  $m$  binary consensus objects and window registers of size at most  $l$  are allocated at initialization
- ▶ A **large but finite** number of processes ( $m! \times (2l)^m$ ) arrive
- ▶ We build an execution where two partitions can never communicate
  - ▶ Values written on window registers are overwritten
  - ▶ All values proposed to binary consensus are the same
- ▶ Finally, two different values are decided



# The Complete Extended Wait-free Hierarchy

Arrival Models		Universal without infinite allocation?				
		Infinite	Finite	Bounded		
Universal with infinite allocation?	Infinite	∞	∞	∞	✓	
	Finite	∞	∞	✓	✓	
	Infinite	∞	✓	✓	✓	
	Finite	∞	✓	✓	✓	
✓	✓	✓	empty	binary consensus $\infty_{1}^3$	binary consensus + iterator stack $\infty_{2}^3$	consensus $\infty_{3}^3$
∞	✓	✓	empty	window registers $\infty_{1}^2$	iterator stack $\infty_{2}^2$	empty $\infty_{3}^2$
∞	∞	✓	empty	empty $\infty_{1}^1$	empty $\infty_{2}^1$	empty $\infty_{3}^1$
∞	∞	∞	Read/Write	(if universal without infinite allocation, still universal with infinite allocation)		

[Herlihy 1991]



# Open Problems

## Relevance of consensus number $\infty_1^3$

- ▶  $k$  binary consensus objects synchronize  $2^k$  processes
- ▶ Are there objects of consensus number  $\infty_1^3$  that do not allow a polylogarithmic implementation of consensus?

## Identification of simpler objects

- ▶ Window registers and iterator stacks are complex objects
  - ▶ Are there equivalent special instructions?
- ▶ We only know a composition for consensus number  $\infty_2^3$ 
  - ▶ Are there not-composed objects of consensus number  $\infty_2^3$ ?
  - ▶ Are there objects of consensus number  $\infty_1^3$  and  $\infty_2^2$  whose composition has consensus number  $\infty_3^3$ ?

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